The International Journal of Engineering and Science (IJES)
ISSN (e): 2319 – 1813 ISSN (p): 23-19 – 1805
|| Pages || PP 54-58 || 2020 ||

Existence and Uniqueness Result of Nonlinear Boundary ValueProblem Arising Due to Hiemenz Flow with Slip Boundary Conditions

Suman Sarkar *[1], Bikash Sahoo[2]
[1]*Department of Mathematics, NIT Rourkela, Odisha, India*
[2]*Department of Mathematics, NIT Rourkela, Odisha, India*
**Corresponding author Email: 517ma1003@nitrkl.ac.in*

--***ABSTRACT***--
In this article, a topological existence and uniqueness proof is given for a non-linear boundary value problem arising due to the steady, laminar stagnation point flow of an incompressible viscous fluid near a rough surface. The no-slip boundary conditions are replaced by the partial slip boundary conditions, owing to the surface roughness. Further, numerical results are also given to validate the qualitative analysis of the solution.

KEYWORDS;*-Stagnation point flow, Existence-Uniqueness, Shooting argument*

I. INTRODUCTION

The Hiemenz flow of a viscous fluid has drawn the attention of many researchers due to its practical and theoretical importance. This is one of the most extensively studied problems. This 2 dimensional stagnation flow is popularly known as the 'Hiemenz flow' due to the pioneering work of K. Hiemenz [1]. The classical Hiemenz flow problem has been extended in many ways including diverse physical effects. One can refer the work of Sahoo [2] and all references therein regarding the stagnation point flow and heat transfer of Newtonian and non-Newtonian fluids. The no-slip boundary conditions are the central tenets of the Navier-Stokes theory. However, some rarefied gases and non-Newtonian fluids exhibit slip boundary conditions. Wang [3] has considered the stagnation point flow with slip boundary conditions, where the amount of relative slip depends on linearly on the local shear stress. He obtained an exact numerical solution of the Navier-Stokes equations. In this paper, we have used a topological shooting argument [4, 5] to study the existence and uniqueness of the resulting third-order nonlinear free boundary value problem with slip boundary conditions.

II. FLOW ANALYSIS

We consider the steady, two-dimensional, in-compressible, viscous fluid normally impinging on a rough wall. The wall coincides with the x-axis, and fluid occupies the domain $y > 0$. The free stream velocity is given by $U_x = ax, U_y = -ay$, a being a constant. The corresponding equations of motion and continuity are given by

$$(\vec{u}.\vec{\nabla})\vec{u} = -\frac{\vec{\nabla}P}{\rho} + \nu\nabla^2\vec{u} \quad (1)$$

$$\vec{\nabla}.\vec{u} = 0, \quad (2)$$

where $\vec{u} = (u_x, u_y)$ is the fluid velocity, ρ is the fluid density and ν is the kinematic viscosity.

Relevant boundary conditions for the rough wall are [3]

$$u_x = \lambda^* \frac{\partial u_x}{\partial y}, u_y = 0 \quad at \quad y = 0$$

$$u_x \to ax \quad as \quad y \to \infty. \quad (3)$$

Here λ^* denotes the slip coefficient. Using the following similarity transformations

$$\vec{u} = axf'(\zeta)\hat{\imath} - \sqrt{\frac{a\mu}{\rho}} f(\zeta)\hat{\jmath} \quad (4)$$

$$\zeta = \sqrt{\frac{a\rho}{\mu}} y \quad (5)$$

and following Wang [3], the equations of motion reduce to

$$f''' + ff'' - f'^2 + 1 = 0. \quad (6)$$

The boundary conditions (3) become

$$f(0) = 0, \quad f'(0) = \lambda f''(0), \quad (7)$$

$$f'(\infty) \to 1.$$

Here $\lambda = \lambda^* (\frac{a}{\nu})^{\frac{1}{2}}$ is the slip parameter.

III. EXISTENCE AND UNIQUENESS RESULTS

In this section, the existence and uniqueness of the solution of Eq. (6) subject to partial slip boundary conditions (7) are discussed. We have to prove the following two theorems:

Theorem 1 *(Existence) For any $\lambda > 0$, the boundary value problem (6)-(7) has a solution and further the solution is monotonic increasing.*

Theorem 2 *(Uniqueness) The solution of the boundary value problem (6)-(7) is unique for $\lambda > 0$.*

We will use the shooting argument to prove the existence and uniqueness result of the solution of the boundary value problem (BVP) (6)-(7). In past, this topological argument has been adopted by many researchers [4, 6, 7] for its advantages such as it proves not only the existence results but also yields considerable information about the solution. In this method, we have to find suitable value of $f''(0)$, so that the solution $f(\zeta)$ satisfies the terminal boundary conditions $f'(\zeta_\infty) \to 1$. Hence, we will study the ordinary differential equation 6 with respect to the following initial conditions

$$f(0) = 0, \; f'(0) = \lambda\alpha, \; f''(0) = \alpha, \quad (8)$$

where $\lambda \geq 0$. Since α is a free variable, the solution is a function of both ζ and α, so we will denote the solution as $f'(\zeta, \alpha)$.

Now, we define A and B subset of $(0, \infty)$

$$A = \{\alpha > 0 : f''(\zeta; \alpha) = 0 \text{ before } \lambda\alpha < f' < 1\}$$
$$B = \{\alpha > 0 : f'(\zeta; \alpha) = 1 \text{ before } 0 < f''(\zeta; \alpha) < \alpha\}$$

The lemmas defined below are used to prove Theorem 1.

Lemma 1 *A and B are disjoint and open.*

Proof : The sets A and B are clearly disjoint. To prove A is open, let us choose a point α in A. Then there exists ζ_0 such that $f''(\zeta_0, \alpha) = 0$ but $\lambda\alpha < f'(\zeta, \alpha) < 1$ on $\zeta \in (0, \zeta_0]$. Now from Eq. (6), $f'''(\zeta_0, \alpha) = f'^2(\zeta_0, \alpha) - 1 = (f'(\zeta_0, \alpha) - 1)(f'(\zeta_0, \alpha) + 1) \neq 0$.

Hence, by continuous solutions of initial value problem with its initial conditions, there exists a neighborhood of α such that for all points in the neighborhood, $f''(\zeta)$ has a root and $\lambda\alpha < f'(\zeta) < 1$. Therefore, A is open. Similarly, it can be proved that the set B is open.

Lemma 2 *If α is sufficiently small, then it is in A.*

Proof : Let us first consider $\alpha = 0$. Since $\lambda > 0$, from Eq. (6) we have $f'''(0; \alpha) = f'^2(0; \alpha) - 1 = \lambda^2\alpha^2 - 1 = (\lambda\alpha - 1)(\lambda\alpha + 1) < 0$. Hence, we have $f''(\zeta; 0) < 0$ and $f'(\zeta; 0) < 1$ in small neighborhood of $\zeta = 0$. Therefore, by continuous solutions of initial value problem with its initial conditions, there exists some $\alpha > 0$ such that $f''(\zeta; \alpha) < 0$ and $f'(\zeta; \alpha) < 1$ for ζ is in neighborhood of $\zeta = 0$. But for any $\alpha > 0$ we have $f''(0; \alpha) = \alpha > 0$. So, there exists a first ζ_0 such that $f''(\zeta_0; \alpha) = 0$ and $f'(\zeta; \alpha) < 1$ for $\zeta \in (0, \zeta_0]$. Thus, when $\alpha > 0$ is sufficiently small, then it is in A.

Lemma 3 *If α is sufficiently large, then it is in B.*

Proof : Integrating Eq. (6) in between 0 to ζ we get,

$$f''(\zeta) - \alpha = \int_0^\zeta (-ff'' + f'^2 - 1)d\tau$$

$$f''(\zeta) = \alpha - \zeta - f(\zeta)f'(\zeta) + 2\int_0^\zeta f'^2(\tau)d\tau. \qquad (9)$$

We shall use the above identity to claim that sufficiently large α is in B i.e., $f'(\zeta;\alpha) = 1$ strictly when $0 < f''(\zeta;\alpha) < \alpha$. First, suppose that the affirmation is false. Then one of the following assertions must occur:

- $f''(\zeta;\alpha) = 0$ at some point ζ where $f'(\zeta;\alpha) < 1$.
- $f''(\zeta;\alpha) > 0$ and $f'(\zeta;\alpha) < 1$ for all ζ.
- $f''(\zeta;\alpha) = 0$ and $f'(\zeta;\alpha) = 1$ simultaneously.

We want to eliminate each of this assertion. To begin with the first, let there exists a first ζ_1 such that $f''(\zeta_1;\alpha) = 0$ with $\lambda\alpha < f'(\zeta;\alpha) < 1$ for $\zeta \in (0,\zeta_1]$. Thus, $f(\zeta;\alpha)$ is bounded and $\lambda\alpha\zeta < f(\zeta;\alpha) < \zeta$ for $\zeta \in (0,\zeta_1]$. Therefore,

$$f''(\zeta;\alpha) \geq \alpha - 2\zeta_1 \ \ for \ \ \zeta \in (0,\zeta_1] \qquad (10)$$

Thus, if we choose sufficiently large α then $f''(\zeta;\alpha) > 0$ for all ζ, which is clearly a contradiction. Again, the second assertion cannot happen for sufficiently large α using a similar argument. Now we are left with the third possibility. $f''(\zeta;\alpha) = 0$ and $f'(\zeta;\alpha) = 1$ happen simultaneously for sufficiently large α. From Eq. (6) we get $f^k(\zeta;\alpha) = 0$ for $k \geq 3$, which implies $f'(\zeta;\alpha) = 1$ as a constant. This is contradicting the basic fact of initial value problem (6) with condition (8), as $f'(0) = \lambda\alpha \neq 1$. Therefore, sufficiently large α is in B.

Remark : The non-empty sets A and B are disjoint subset of $(0,\infty)$ and both are also open from lemmas (1)-(3).

Proof of Theorem 1: The set $(0,\infty)$ is connected and hence $A \cup B \neq (0,\infty)$ by previous remark. Thus, there exists α^* which is neither in A nor B. Also in Lemma (3) we observe that $f''(\zeta;\alpha^*) = 0$ and $f'(\zeta;\alpha^*) = 1$ can not happen simultaneously. Therefore, the only one possibility is $f''(\zeta;\alpha) > 0$ and $\lambda\alpha^* < f'(\zeta;\alpha^*) < 1$. From Eq. (6) we have $f'(\infty;\alpha) \to 1$, giving the existence of monotonic increasing solution of boundary value problem (6)-(7).

Proof of Theorem 2: We claim that there exists a unique α for the boundary value problem. For sake of contradiction suppose that two values of α exist and which satisfy all the boundary conditions. The solutions are denoted as $f'(\zeta;\alpha_1)$ and $f'(\zeta;\alpha_2)$ (choose $\alpha_2 > \alpha_1$). Applying mean value theorem in the interval $[\alpha_1,\alpha_2]$, we have

$$f'(\zeta;\alpha_2) - f'(\zeta;\alpha_1) = \left(\frac{\partial f'}{\partial\alpha}\right)_{\alpha=\alpha'}(\alpha_2 - \alpha_1), \qquad (11)$$

where $\alpha' \in [\alpha_1,\alpha_2]$. Thus, for $\zeta \to \infty$,

$$\left(\frac{\partial f'}{\partial\alpha}\right)(\infty,\alpha') = f'(\infty,\alpha_2) - f'(\infty,\alpha_1) = 0. \quad (12)$$

Now suppose that $\frac{\partial f'}{\partial\alpha} = v'(\zeta;\alpha)$ and differentiating Eq. (6) and it's initial conditions (8) with respect to α we have,

$$v'''(\zeta;\alpha) + v(\zeta;\alpha)f''(\zeta;\alpha) + f(\zeta;\alpha)v''(\zeta;\alpha) - 2f'(\zeta;\alpha)v'(\zeta;\alpha) = 0 \qquad (13)$$

and

$$v(0;\alpha) = 0, v'(0;\alpha) = \lambda, v''(0;\alpha) = 1, v'''(0;\alpha) = 2\lambda^2\alpha, \qquad (14)$$

where prime denotes the differentiation with respect to ζ. Again, differentiating Eq. (13) w.r.t ζ, we get

$$v^{iv}(\zeta;\alpha) = -v(\zeta;\alpha)f'''(\zeta;\alpha) + v'(\zeta;\alpha)f''(\zeta;\alpha) + f'(\zeta;\alpha)v''(\zeta;\alpha)f(\zeta;\alpha)v'''(\zeta;\alpha) \quad (15)$$

Therefore,

$$v^{iv}(0;\alpha) = 2\lambda\alpha > 0 \qquad (16)$$

Thus, from the initial condition (??) on v we have $v'(\zeta;\alpha) > 0, v''(\zeta;\alpha) > 1$ and $v'''(\zeta;\alpha) > 0$ when $0 < \zeta < \epsilon$ for some $\epsilon > 0$. In particular, positive $v'(\zeta;\alpha)$ is concave up increasing initially and to become zero, it has to change first from concave up to concave down. Thus there exists a first ζ_2 such that $v'''(\zeta_2;\alpha) = 0$ and $v^{iv}(\zeta_2;\alpha) \leq 0$. But until this point ζ_2, $v(\zeta;\alpha)$ and all its derivatives through $v'''(\zeta;\alpha)$ are positive and increasing. Therefore, $f(\zeta;\alpha)$ and all its derivative through $f'''(\zeta;\alpha)$ are increasing function with respect to α. Thus, for $\alpha \in [\alpha_1,\alpha_2]$ we have

$$v^{iv}(\zeta_2;\alpha) = -v(\zeta_2;\alpha)f'''(\zeta_2;\alpha) + v'(\zeta_2;\alpha)f''(\zeta_2;\alpha) + f'(\zeta_2;\alpha)v''(\zeta_2;\alpha) \qquad (17)$$

which implies $v^{iv}(\zeta_2;\alpha) > 0$ and it is clearly a contradiction. Thus $v'(\zeta;\alpha)$ can never become zero contradicting the fact (12). Hence, we conclude that there exists unique α which satisfy all the boundary conditions.

Remark : The solution of boundary value problem (6)-(7) exists and it is unique. Moreover, the solution is monotonic increasing.

IV. NUMERICAL RESULTS

In this section, the resulting free boundary value problem (6), subject to boundary conditions (7) has been integrated using shooting method, which is a combination of fourth order Runge-Kutta method and a secant method. The numerical infinity (ζ_∞) is taken large enough and is kept unchanged throughout the program. The effects of slip parameter on the velocity components are shown in Figs. 1 and 2. Near the surface(i.e., at y=0), the velocity component, $f'(\zeta)$ increases with an increase in the slip parameter λ and assumes its asymptotic value near to the surface. Thus, slip decreases the boundary layer thickness. The values of the missing initial guess, $f''(0)$ for different values of the slip parameter are tabulated in Table 1. Physically, $f''(0)$ is the value of the skin friction coefficient. It is clear that the skin friction coefficient decreases with an increase in slip and the flow behaves like an in-viscid flow for high values of λ. These findings are in agreement with the results of Wang [3].

$\boldsymbol{\lambda}$	**0**	**0.5**	**1**	**2**	**5**
$f''(0)$	1.232588	0.821479	0.593462	0.375886	0.177260

Table 1: Variations of missing initial conditions α with different value of slip parameter λ.

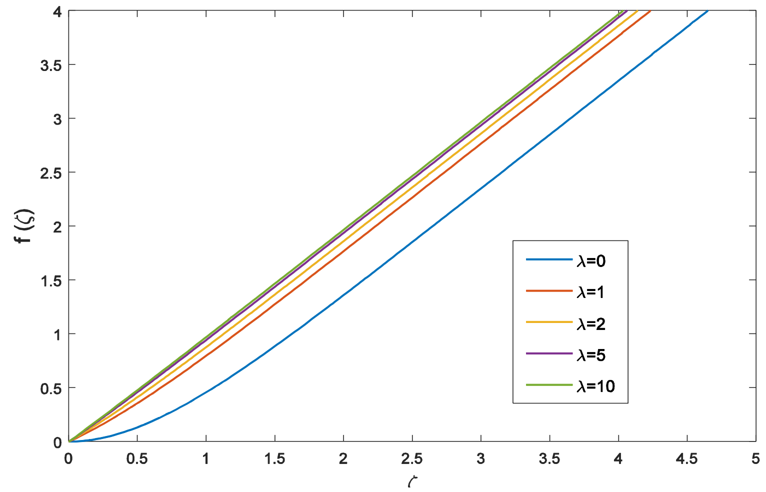

Figure 1: The similarity solution$f(\zeta)$ at differentλ.

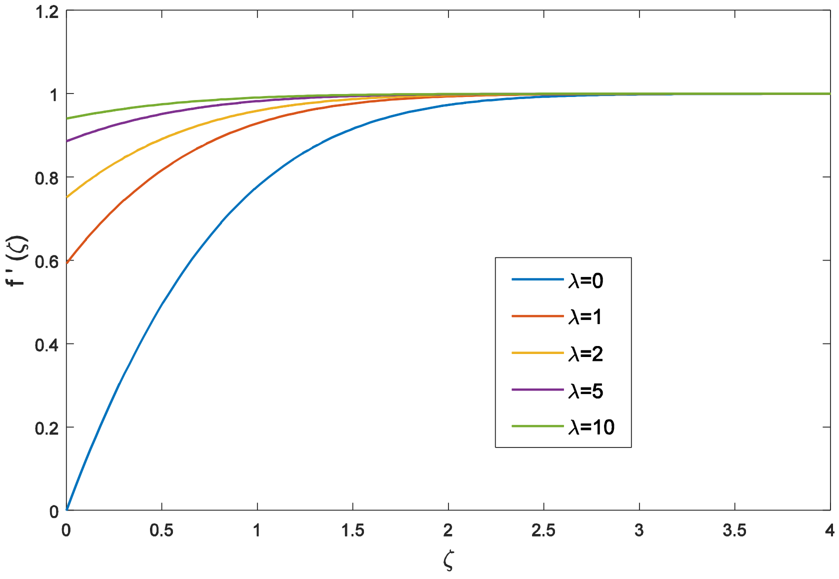

Figure 2: Velocity profiles for different values of λ.

V. CONCLUTIONS

In this paper, the existence and uniqueness results are established for a non-linear third order differential equation, arising due to the two-dimensional stagnation point flow over a flat horizontal rough surface. Using topological shooting argument, we proved that there is a unique monotonic increasing solution to the nonlinear ordinary differential equation, subject to the slip boundary conditions. Also, it has been seen that the solutions $f(\zeta)$ have the following properties:

1. $f(\zeta)$ is monotonically increasing and non-negative;
2. $f'(\zeta)$ is also monotonically increasing, non-negative and bounded;
3. $f''(\zeta)$ is monotonically decreasing, positive and bounded.

Further, our qualitative analysis are supported by obtained numerical results and it also agrees with the results of Wang [3].

REFERENCE

[1]. K Hiemenz. Die grenzschicht an einem in den gleichformigen flussigkeitsstrom eingetauchten geraden kreiszylinder. *Dinglers Polytech. J.*, 326:321–324, 1911.

[2]. B. Sahoo. Hiemenz flow and heat transfer of a third grade fluid. *Communications in Nonlinear Science and Numerical Simulation*, 14(3):811–826, 2009.

[3]. CY. Wang. Stagnation flows with slip: exact solutions of the navier-stokes equations. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 54(1):184–189, 2003.

[4]. J. Paullet and P. Weidman. Analysis of stagnation point flow toward a stretching sheet. *International Journal of Non-Linear Mechanics*, 42(9):1084–1091, 2007.

[5]. J. Paullet. Analysis of stagnation point flow of an upper-convected Maxwell fluid. *Electronic Journal of Differential Equations*, 302:1-14, 2017.

[6]. O. Padé. On the solution of falkner–skan equations. *Journal of mathematical analysis and applications*, 285(1):264–274, 2003.

[7]. [M Miklavčič and CY Wang. The flow due to a rough rotating disk. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 55(2):235–246, 2004.